Cartan-Kähler Theory and Applicationsto Local Isometric and Conformal Embedding

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Abstract

The goal of this lecture is to give a brief introduction to Cartan-Kähler's theory. As examples to the application of this theory, we choose the local isometric and conformal embedding. We provide lots of details and explanations of the calculation and the tools used[1].

1 Cartan's Structure Equations

Let $\xi = (E, \pi, M)$ be a vector bundle. Denote $(\mathfrak{X}(M), [,])$ the Lie algebra of vector fields on M and $\Gamma(E)$ the moduli space of cross-sections of the vector bundle E.

1.1 Connection on a vector bundle

A connection on a vector bundle E is a choice of complement of vertical vector fields on E. A connection induces a covariant differential operator ∇ on E. A covariant derivative ∇ on a vector bundle E is a way to "differentiate" bundle sections along tangent vectors and it is sometimes called a connection.

Definition 1.1.1. *A connection on a vector bundle E is an linear operator defined as follows:*

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \Gamma(E) &\longrightarrow \Gamma(E) \\ (X, S) &\longmapsto \nabla_X S \end{aligned}$$

[1]See the Master Thesis [7] on which the lecture is based

satisfying

$$\nabla_{(X_1+X_2)}S = \nabla_{X_1}S + \nabla_{X_2}S, \qquad \nabla_{(fX)}S = f\nabla_X S$$
$$\nabla_X(S_1+S_2) = \nabla_X S_1 + \nabla_X S_2, \qquad \nabla_X(fS) = X(f)S + f\nabla_X S$$

$\forall X, X_1, X_2, Y, Y_1, Y_2 \in \mathfrak{X}(M)$ *and* $\forall S, S_1, S_2 \in \Gamma(E)$.

1.1.1 Curvature of a Connection

Definition 1.1.2. *The curvature of a connection ∇ is a vector valued 2-form*

$$\begin{aligned} \mathcal{R} : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) &\longrightarrow \Gamma(E) \\ X, Y, S &\longmapsto \mathcal{R}(X,Y)S \end{aligned}$$

defined by $\mathcal{R}(X,Y)S = \Big([\nabla_X,\ \nabla_Y] - \nabla_{[X,\ Y]}\Big)S$

Theorem 1.1. *For any f, g and h smooth functions on M, $S \in \Gamma(E)$ a section of ξ and $X, Y \in \mathfrak{X}(M)$ two tangent vector fields of M, we have*

$$\mathcal{R}(fX, gY)(hS) = fgh\mathcal{R}(X,Y)S \tag{1}$$

1.1.2 Connection and Curvature Forms

Let $\xi = (E, \pi, M)$ be a vector bundle over a smooth manifold M with an r-dimensional vector space E as a standard fiber. Let ∇ be a connection on ξ and $\mathcal{R}$ its curvature. We deonte by $\mathcal{U}$ an open set of M.

Definition 1.1.3. *A set of r local sections $S = (S_1, S_2, \ldots, S_r)$ of ξ is called a frame field (or a moving frame) if $\forall p \in \mathcal{U}$, $S(p) = \Big(S_1(p), S_2(p), \ldots, S_r(p)\Big)$ form a basis of the fiber E_p over p.*

Let $S = (S_1, S_2, \ldots, S_r)$ be a frame field,∇ a connection on ξ and $X \in \mathfrak{X}(M)$ a tangent vector field on M. Then $\nabla_X S_j$ is another section of ξ and it can be expressed in the frame field S as follows:

$$\nabla_X S_j = \sum_{i=1}^{r} \omega_{ij}(X) S_i \tag{2}$$

where $\omega_{ij} \in \mathcal{A}^1(M)$ are differential 1-forms[2] on M and $\omega_{ij}(X)$ are smooth functions on M.

[2] $\mathcal{A}^k(M)$ denote the set of differential k-form on M (we choose this notation instead of the standard notation $\Omega^k(M)$ to not mix with the curvature form.

Definition 1.1.4. *The $r \times r$ matrix $\omega = (\omega_{ij})$ is called the connection 1-form of ∇.*

The connection ∇ is completely determined by the matrix $\omega = (\omega_{ij})$. Conversely, a matrix of differential 1-forms on M determines a connection (in a non-invariant way depending on the choice of the moving frame).

Let $X, Y \in \mathfrak{X}(M)$ two tangent vector fields. Then $\mathcal{R}(X,Y)S_j$ are sections of ξ., and can be expressed on the frame field S as follows:

$$\mathcal{R}(X,Y)S_j = \sum_{i=1}^{r} \Omega_{ij}(X,Y)S_i \tag{3}$$

where $\Omega_{ij} \in \mathcal{A}^2(M)$ are differential 2-forms on M and $\Omega_{ij}(X,Y)$ are smooth functions on M.

Definition 1.1.5. *The $r \times r$matrix $\Omega = (\Omega_{ij})$ whoose entries are differential 2-forms, is called the curvature 2-form of the connection ∇.*

We state the following theorem[3] that gives the relation between the connection 1-form ω and the curvature 2-form Ω.

Theorem 1.2.

$$d\omega + \omega \wedge \omega = \Omega \quad \textit{(matrix form)} \tag{4}$$

or

$$d\omega_{ij} + \sum_{k=1}^{m} \omega_{ik} \wedge \omega_{kj} = \Omega_{ij} \quad \textit{(on components)} \tag{5}$$

1.2 The Induced Connection

Let $\xi = (E, \pi, M)$ and $\xi' = (E', \pi', M)$ be two vector bundles on M. Consider a map $f : M \longrightarrow M$ and denote $\tilde{f} : E \longrightarrow E'$ the associated bundle map i.e. $(f, \tilde{f})$ satisfies the following commutative diagramme:

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "f"'] & M \end{tikzcd}$$

[3] In the tangent bundle case, this theorem gives Cartan's second equation, as we will see later.

If ∇' is a connection on E', the vector bundle morphism induces a pull-back connection on E

$$\nabla = \tilde{f}^*\nabla' \tag{6}$$

such that for any $S' \in \Gamma(E')$ and $X \in \mathfrak{X}(M)$, $\nabla_X(\tilde{f}^*S') = (\tilde{f}^*\nabla')_X(\tilde{f}^*S') = \tilde{f}^*\Big(\nabla_{f_*X}S'\Big)$ where $f_{*,p} : T_pM \longrightarrow T_{f(p)}M$ is the linear tangent map .

We can also induce a connection on ξ by another way. The connection ∇' is completely determined by the matrix of differential 1-forms $\omega' = (\omega'_{ij})$, and we define ∇ by the matrix ω whose entries ω_{ij} are the pull-back of ω'_{ij} by$\tilde{f}$, i.e. $\omega = \tilde{f}^*\omega'$.

The pull back commute with the exterior differentiation and with the exterior product[4], so, the curvature 2-form Ω of ∇ is the pull back of the curvature 2-form of ∇', i.e. $\Omega = \tilde{f}^*\Omega'$.

1.3 Metric Connection

Let $\xi = (E, \pi, M)$ be a vector bundle. We denote by ∇ a connection on ξ determined by a matrix of 1-forms ω. Let Ω be the associated curvature 2-form and g a Riemannian metric on ξ (i.e. a positively-defined scalar product on each fiber).

Definition 1.3.1. ∇ *is a connection on* ξ *compatible with the metric* g *(or a metric connection) if* ∇ *satisfies to the following property (Leibniz's identity):*

$$\begin{aligned} &\nabla_X\Big(g(S_1, S_2)\Big) = g(\nabla_X S_1, S_2) + g(S_1 \nabla_X S_2) \\ &\forall S_1, S_2 \in \Gamma(E), \ \textit{and} \ \forall X \in \mathfrak{X}(M) \end{aligned} \tag{7}$$

Proposition 1.3.1. *Let* $S = (S_1, S_2, \ldots, S_n)$ *be an orthonormal frame field with respect to* g*, i.e.* $g_p(S_i, S_j) = \delta_{ij}$ *for all* $p \in \mathcal{U}$*,* $i, j = 1, \ldots, r$*, then the matrix of 1-forms* ω *associated to* S *and the curvature matrix of 2-form are both skew-symmetric.*

[4] $d(f^*\alpha) = f^*(d\alpha)$ and $f^*(\alpha \wedge \beta) = f^*(\alpha) \wedge f^*(\beta)$ for all $\alpha, \beta \in \mathcal{A}(M)$.

1.4 Tangent Bundle Case

1.4.1 Torsion of a Connexion on a Tangent Bundle

Let us consider now, a local frame field $S = (S_1, \ldots, S_m)$ over $\mathcal{U} \subset M$ where $S_i \in \mathfrak{X}(\mathcal{U})$ are local tangent vectors fields (i.e. local sections of the tangent bundle such that for all $p \in \mathcal{U}$, $\Big(S_1(p), \ldots, S_m(p)\Big)$ forms a basis of the tangent vector space of M).

Definition 1.4.1. *If S is a local orthonormal frame field, the associated coframe field $\eta = \Big(\eta_1, \ldots, \eta_m\Big)$ is a local frame field of 1-forms, such that for all $p \in \mathcal{U}$, $\eta_i(p)\Big(S_j\Big) = \delta_{ij}$.*

We define then a differential 2-form Θ as follows:

$$d\eta + \omega \wedge \eta = \Theta \tag{8}$$

Definition 1.4.2. Θ *is called the torsion 2-form of* ∇.

Proposition 1.4.1. *On a tangent bundle, the four forms η, ω, Θ and Ω are connected by the folowing equations*

$$d\Theta + \omega \wedge \Theta = \Omega \wedge \eta \tag{9}$$

and

$$d\Omega = \Omega \wedge \omega - \omega \wedge \Omega \tag{10}$$

The equation (10) is the expression of the Bianchi identity via the connection 1-form and the curvature 2-form. Equation (10) is also valid on a arbitrary vector bundle.

1.4.2 Cartan's Structure Equations

Let (M, g) be an m-dimensional Riemannian manifold. Let $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ an orthonormed coframe field on M ($\eta_j \in \mathcal{A}^1(M)$). According to equations (8), (5) and the proposition 1.3.1, we establish the Cartan structure equations:

$$\begin{cases} d\eta_i + \displaystyle\sum_{j=1}^{m} \omega_{ij} \wedge \eta_j = 0 \qquad \text{(torsion-free)} \\ d\omega_{ij} + \displaystyle\sum_{k=1}^{m} \omega_{ik} \wedge \omega_{kj} = \Omega_{ij} \end{cases} \tag{11}$$

where the matrix (ω_{ij}) is the Lévi-Civita connection 1-form (free torsion connection which is compatible with the riemannian metric g). Because η is an orthonormed coframe field, (ω_{ij}) is skew-symmetric (proposition 1.3.1.). (Ω_{ij}) is the curvature 2-form matrix of the riemannian connection ($\Omega_{ij} = \frac{1}{2}\sum_{k,l=1}^{m} \mathcal{R}_{ijkl}\eta_k \wedge \eta_l$).

1.5 The Cartan Lemma

We end this section with a technical lemma, which is easy to establish and at the same time rich applications . This lemma will not only be useful for isometric embedding problem, but also for many calculus in differential geometry.

Lemma 1.5.1. *Let M be an m-dimensional manifold. $\omega_1, \omega_2, \dots, \omega_r$ a set of linearly independent differential $1-$forms ($r \leq n$) and $\theta_1, \theta_2, \dots, \theta_r$ differential $1-$forms such that*

$$\sum_{i=1}^{r} \theta_i \wedge \omega_i = 0 \tag{12}$$

then there exists r^2 functions h_{ij} in $\mathcal{C}^1(M)$ such that

$$\theta_i = \sum_{j=1}^{r} h_{ij}\omega_j \qquad \text{with } h_{ij} = h_{ji}. \tag{13}$$

2 Exterior Differential Systems and Ideals

2.1 Exterior Differential Systems

Denote $\mathcal{A}(M)$ the space of smooth differential forms[5] on M.

Definition 2.1.1. *An exterior differential system is a finite set of differential forms $I = \{\omega_1, \omega_2, \dots, \omega_k\} \subset \mathcal{A}(M)$ for which there is a set of equations $\{\omega_i = 0 | \omega_i \in I\}$.*
such that one can write the exterior differential system as follow:

$$\begin{cases} \omega_1 = 0 \\ \omega_2 = 0 \\ \vdots \\ \omega_k = 0 \end{cases}$$

[5]This is a graded algebra under the wedge product.

Definition 2.1.2. *An exterior differential system $I \subset \mathcal{A}(M)$ is said to be Pffafian if I contains only differential 1-forms, i.e. $I \subset \mathcal{A}^1(M)$.*

2.2 Exterior Ideals

Definition 2.2.1. *Let $\mathcal{I} \subset \mathcal{A}(M)$ a set of differentiable forms. $\mathcal{I}$ is an exterior ideal if:*

1. *The exterior product of any differential form of $\mathcal{I}$ by a differential form of $\mathcal{A}(M)$ belong to $\mathcal{I}$.*
2. *The sum of any two differential forms of the same degree belonging to $\mathcal{I}$, belong also to $\mathcal{I}$.*

Definition 2.2.2. *Let $I \subset \mathcal{A}(M)$ an exterior differential system. The exterior ideal generated by I is the smallest exterior ideal containing I.*

2.3 Exterior Differential Ideals

Definition 2.3.1. *Let $\mathcal{I} \subset \mathcal{A}(M)$ a set of differential forms. $\mathcal{I}$ is an exterior differential ideal if $\mathcal{I}$ is an exterior ideal closed under the exterior differentiation, i.e. $\forall \omega \in \mathcal{I}, d\omega \in \mathcal{I}$ (we can also write $d\mathcal{I} \subset \mathcal{I}$).*

Definition 2.3.2. *Let $I \subset \mathcal{A}(M)$ an exterior differential system. The exterior differential ideal generated by I is the smallest exterior differential ideal containing I.*

2.4 Closed Exterior Differential Systems

Definition 2.4.1. *An exterior differential system $I \subset \mathcal{A}(M)$ is said closed if the exterior differentiation of any form of I, belong to the exterior ideal generated by I.*

Proposition 2.4.1. *An exterior differential system I is closed if and only if the exterior differential ideal generated by I is equal to the exterior ideal generated by I. In particular, $I \cup dI$ is closed .*

2.5 Solutions of an Exterior Differential System

Definition 2.5.1. *Let $I \subset \mathcal{A}(M)$ be an exterior differential system and N a sub-manifold of M. N is an integral manifold of I if $i^*\omega = 0, \forall \omega \in I$, where i is an embedding $i : N \longrightarrow M$.*

3 Introduction to Cartan-Kähler Theory

We consider in this section, an m-dimensional real manifold M and $\mathcal{I} \subset \mathcal{A}(M)$ an exterior differential ideal on M.

3.1 Integral Elements

Definition 3.1.1. *Let $z \in M$ and $E \subset T_zM$ a linear subspace of T_zM. E is an integral element of $\mathcal{I}$ if $\varphi_E = 0$ for all $\varphi \in \mathcal{I}$. We denote by $\mathcal{V}_p(\mathcal{I})$ the set of p-dimensional integral elements of $\mathcal{I}$.*

Definition 3.1.2. *N is an integral manifold of $\mathcal{I}$ if and only if each tangent space of N is an integral element of $\mathcal{I}$.*

Proposition 3.1.1. *If E is a p-dimensional integral element of $\mathcal{I}$, then every subspace of E are also integral elements of $\mathcal{I}$.*

We denote by $\mathcal{I}_p = \mathcal{I} \cap \mathcal{A}^p(M)$ the set of differential p-forms of $\mathcal{I}$.

Proposition 3.1.2. $\mathcal{V}_p(\mathcal{I}) = \{E \in G_p(TM) | \varphi_E = 0 \textit{ for all } \varphi \in \mathcal{I}_p\}$

Definition 3.1.3. *Let E an integral element of $\mathcal{I}$. Let $\{e_1, e_2, \ldots, e_p\}$ a basis of $E \subset T_zM$. The polar space of E, denoted by $H(E)$, is the vector space defined as follow:*

$$H(E) = \{v \in T_zM | \varphi(v, e_1, e_2, \ldots, e_p) = 0 \textit{ for all } \varphi \in \mathcal{I}_{p+1}\}. \tag{14}$$

Notice that $E \subset H(E)$. This implies that a differential form is alternate. The polar space plays an important role in exterior differential system theory as we shall see in the following proposition.

Proposition 3.1.3. *Let $E \in \mathcal{V}_p(\mathcal{I})$ be an p-dimensional integral element of $\mathcal{I}$. A $(p+1)$-dimensional vector space $E^+ \subset T_zM$ which contains E is an integral element of $\mathcal{I}$ if and only if $E^+ \subset H(E)$.*

In order to check if a given p-dimensional integral element of an exterior differential ideal $\mathcal{I}$ is contained in a $(p+1)$-dimensional integral element of $\mathcal{I}$, we introduce the following function $r : \mathcal{V}_p(\mathcal{I}) \longrightarrow \mathbb{Z}$, $r(E) = dim H(E) - (p+1)$ is a relative integer, $\forall E \in \mathcal{V}_p(\mathcal{I})$.

Notice that $r(E) \geq 1$. If $r(E) = -1$, then E is contained in any $(p+1)$-dimensional integral element of $\mathcal{I}$.

3.1.1 Kähler-Ordinary and Kähler-Regular Integral Elements

Let Δ a differential n-form on a m-dimensional manifold M. Let[6] $G_n(TM,\Delta) = \{E \in G_n(TM)/\Delta_E \neq 0\}$. We denote by $\mathcal{V}_n(\mathcal{I},\Delta) = \mathcal{V}_n(\mathcal{I}) \cap G_n(TM,\Delta)$ the set of integral elements of $\mathcal{I}$ on which $\Delta_E \neq 0$.

Definition 3.1.4. *An integral element $E \in \mathcal{V}_n(\mathcal{I})$ is called Kähler-ordinary if there exists a differential n-form Δ such that $\Delta_E \neq 0$. Moreover, if the function r is locally constant in some neighborhood of E, then E is said Kähler-regular.*

3.1.2 Integral Flags, Ordinary and Regular Integral Elements

Definition 3.1.5. *An integral flag of $\mathcal{I}$ on $z \in M$ of length n is a sequence of integral elements E_k of $\mathcal{I}$: $(0)_z \subset E_1 \subset E_2 \subset \cdots \subset E_n \subset T_zM$.*

Definition 3.1.6. *Let I be an exterior differential system on M. An integral element $E \in \mathcal{V}(I)$ is said ordinary if its base point $z \in M$ is an ordinary zero of $I_0 = I \cap \mathcal{A}^0(M)$ and if there exists an integral flag $(0)_z \subset E_1 \subset E_2 \subset \cdots \subset E_n = E \subset T_zM$ where the E_k, $k = 1,\ldots,(n-1)$ are Kähler-regular integral elements. Moreover, if E is Kähler-regular, then E is said regular.*

3.2 Cartan's Test

Theorem 3.1. *(Cartan's test)*
Let $\mathcal{I} \subset \mathcal{A}^*(M)$ be an exterior ideal which does not contain 0-forms (functions on M). Let $(0)_z \subset E_1 \subset E_2 \subset \cdots \subset E_n \subset T_zM$ be an integral flag of $\mathcal{I}$. For any $k < n$, we denote by c_k the codimension of the polar space $H(E_k)$ in T_zM. Then $\mathcal{V}_n(\mathcal{I}) \subset G_n(TM)$ is at least of $c_0 + c_1 + \cdots + c_{n-1}$ codimension at E_n. Moreover, E_n is an ordinary integral flag if and only if E_n has a neighborhood U in $G_n(TM)$ such that $\mathcal{V}_n(\mathcal{I}) \cap U$ is a manifold of $c_0+c_1+\cdots+c_{n-1}$ codimension in U.

Proof. See [1], page 74. □

Proposition 3.2.1. *Let $\mathcal{I} \cap \mathcal{A}^*(M)$ an exterior ideal which do not contains 0-forms. Let $E \in \mathcal{V}_n(\mathcal{I})$ be an integral element of $\mathcal{I}$ at the point $z \in M$. Let $\omega_1,\omega_2,\ldots,\omega_n,\pi_1,\pi_2,\ldots,\pi_s$ (where $s = \dim M - n$) be a coframe in a open neighborhood of $z \in M$ such that $E = \{v \in T_zM/\pi_a(v) = 0$ for all $a = 1,\ldots s\}$. For all $p \leq n$, we define $E_p = \{v \in E|\omega_k(v) = 0$ for all $k > p\}$. Let $\{\varphi_1,\varphi_2,\ldots,\varphi_r\}$ be the set of differential forms which generate the exterior*

[6] $G_n(TM)$ is the Grassmanian of TM, i.e. the set of n-dimensional subspace of TM.

ideal $\mathcal{I}$, where φ_ρ is of $(d_\rho + 1)$ degree.
For all ρ, there exists an expansion

$$\varphi_\rho = \sum_{|J|=d_\rho} \pi_\rho^J \wedge \omega_J + \tilde{\varphi}_\rho \tag{15}$$

where the 1-forms π_ρ^J are linear combinations of the forms π and the terms $\tilde{\varphi}_\rho$ are, either of degree 2 or more on π, or vanish at z.
moreover, we have

$$H(E_p) = \{v \in T_z M | \pi_\rho^J(v) = 0 \text{ for all } \rho \text{ and } \sup J \leq p\} \tag{16}$$

In particular, for the integral flag $(0)_z \subset E_1 \subset E_2 \subset \cdots \subset E_n \cap T_z M$ de $\mathcal{I}$, c_p correspond to the number of linear independent forms $\{\pi_\rho^J|_z$ such that $\sup J \leq p\}$.

Proof. See [1], page 80. □

3.3 Cartan-Kähler's Theorem

The following theorem is a generalization of the well-known Frobenius's theorem.

Theorem 3.2. *(Cartan-Kähler)*
Let $\mathcal{I} \subset \mathcal{A}^*(M)$ be a real analytic exterior differential ideal. Let $P \subset M$ a p-dimensional connected real analytic Kähler-Regular integral manifold of $\mathcal{I}$. Suppose that $r = r(P) \geq 0$. Let $R \subset M$ be a real analytic submanifold of M of codimension r which contains P and such that $T_x R$ and $H(T_x P)$ are transversals in $T_x M$ for all $x \in P \subset M$.
There exists a $(p+1)$-dimensional connected real analytic integral manifold X of $\mathcal{I}$, such that $P \subset X \subset R$. X is unique in the sense that another integral manifold of $\mathcal{I}$ having the stated properties, coincides with X on a open neighborhood of P.

Proof. See [1], page 82. □

The analicity condition of the exterior differential ideal is crucial because of the requirements in the Cauchy-Kowalewski theorem used in the proof of the Cartan-Kähler theorem.

Cartan-Kähler's theorem has an important corollary. Actually, this corollary is often more used than the theorem and it is sometimes called *the Cartan-Kähler theorem.*

Corollary 3.3.1. *(Cartan-Kähler)*
Let $\mathcal{I}$ be an analytic exterior differential ideal on a manifold M. If $E \subset T_zM$ is an ordinary integral element of $\mathcal{I}$, there exists an integral manifold of $\mathcal{I}$ passing through z and having E as a tangent space at the point z.

4 Local Isometric Embedding Problem

We shall state and prove the Burstin-Cartan-Janet-Schlaefli's theorem concerning local isometric embedding of a real analytic Riemannian manifold. The names of the mathematicians are given in alphabetic order. Schlaefli in his paper in 1871 [8] conjectured that an m-dimensional Riemannian manifold can always be, locally, embedded in an $N = \frac{1}{2}m(m+1)$ dimensional Euclidean space. In 1926, Janet [6] proved the result for the dimension 2 by resolving a differential system and explain how we get the result in the general case. In 1927, Élie Cartan [3] gave the complete proof of the result. His method is based on his theory of involutive Pfaffian system. Later in 1931, Burstin [2] generalized Janet's method and obtained the result in the general case.

The proof that we shall give is inspired by Cartan'paper [3], the Bryant, Chern, Gardner, Goldscmidt et Griffiths's book [1] and the Griffiths et Jensen's book [4].

4.1 The Burstin-Cartan-Janet-Schlaefli theorem

Theorem 4.1. *(Burstin 1931-Cartan 1927-Janet 1926-Schlaefli 1871)*
Every m-dimensional real analytic Riemannian manifold can be locally embedded isometrically in an $\dfrac{m(m+1)}{2}$-dimensional Euclidean space.

4.2 Proof of Burstin-Cartan-Janet-Schlaefli's theorem

Steps of the proof of theorem4.1.

1. We shall write down the Cartan structure equations for an m-dimensional real analytic Riemannian manifold M.

2. We shall define a subbundle $\mathcal{F}_m(\mathbb{E}^N)$ of the bundle $\mathcal{F}(\mathbb{E}^N)$ of the Euclidean space $\mathbb{E}^N$, then shall write down the Cartan structure equations for the subbundle $\mathcal{F}_m(\mathbb{E}^N)$.

3. Given an exterior differential system I_0 on $M \times \mathcal{F}_m(\mathbb{E}^N)$, which is not close, we shall prove Claim 4.2.2, which proves that the existence of a

local isometric embedding of M is the existence of an m-dimensional integral manifold of I_0.

4. We will extend this differential system to obtain a closed one. In the process of extension, we will get new equations (the Gauss equation (equ. 29)). We will also show that a closed exterior differential system $\tilde{I}$ with fewer 1-forms than I, will generate the same differential ideal that the one generated by I if the Gauss's equation is satisfied.

5. We establish the lemma 4.2.1., that ensure that the Gauss equations is a surjective submersion. We shall obtain a submanifold with a known dimension.

6. Given the closed exterior differential ideal,, we shall prove the existence of an ordinary integral element by using claim 3.2.1 and the Cartan test. Finally,the Cartan-Kahler theorem ensure then the existence of an integral manifold and lead us to conclude..

Step 1:

Let (M, g) be an m-dimensional real analytic Riemannian manifold, where g is a Riemannian metric, i.e. a covariant symmetric positive defined 2-tensor, such that at a given point p of M, g_p in a orthonormed basis reduce to the identity matrix. However in a open neighborhood of p, the matrix of g can not always be the identity but it can always be reduced to diagonal matrix:

$$g = g_{11}dx^1 \otimes dx^1 + g_{22}dx^2 \otimes dx^2 + \cdots + g_{mm}dx^m \otimes dx^m \tag{17}$$

where the terms g_{ii} are positive functions such that $g_{ii} = 1$ at p. We denote than $\eta_i = \sqrt{g_{ii}}dx^i$. g can be written as follows:

$$g = \eta_1 \otimes \eta_1 + \eta_2 \otimes \eta_2 + \cdots + \eta_m \otimes \eta_m \tag{18}$$

$\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ is than a orthonormal coframe in the neighborhood of $p \in M$. We can establish the Cartan's structure equations:

Cartan's structure equations on M:

$$\begin{cases} d\eta_i + \displaystyle\sum_{j=1}^{m} \eta_{ij} \wedge \eta_j = 0 & \text{(torsion-free)} \\ d\eta_{ij} + \displaystyle\sum_{k=1}^{m} \eta_{ik} \wedge \eta_{kj} = \Omega_{ij} & \end{cases} \tag{19}$$

where (η_{ij}) is the matrix of 1-form of the Lévi-Civita's connection on M (a torsion-free connection compatible with the metric g). Ω_{ij} is the curvature 2-form of the connection.

Step 2:

Let $\mathbb{E}^N$ be an N-dimensional Euclidean space (for the moment, $N > m$) endowed with the usual scalar product ε_N. Let us consider $\mathcal{F}(\mathbb{E}^N)$ a positively-oriented orthonormal frame bundle on $\mathbb{E}^N$. In what follows, we will not work on the entire bundle $\mathcal{F}(\mathbb{E}^N)$,but on a quotient. An element in $\mathcal{F}_m(\mathbb{E}^N)$ has the form $(x; e_1, e_2, \ldots, e_m)$, where $x \in \mathbb{E}^N$ and $(e_1, e_2, \ldots, e_m)$ is a positively-oriented orthonormal set of vectors in $\mathbb{E}^N$. We can consider $\mathcal{F}_m(\mathbb{E}^N)$ as follows: among all the positively-oriented orthonormal frames of $\mathcal{F}(\mathbb{E}^N)$, we take the frames such that the first m elements form a positively-oriented orthonormal set of vectors, then we take the m first vectors of theses frames. So, $\mathcal{F}_m(\mathbb{E}^N)$ is diffeomorphic to $\mathbb{E}^N \times \dfrac{SO(N)}{SO(N-m)}$.

Proposition 4.2.1.

$$dim \mathcal{F}_m(\mathbb{E}^N) = N(m+1) - \frac{m(m+1)}{2} \tag{20}$$

We define on $\mathcal{F}(\mathbb{E}^N)$ a set of 1-forms as follows[7]:

$$\omega_\mu = e_\mu dx \quad \text{ and } \quad \omega_{\mu\nu} = e_\mu de_\nu = -e_\nu de_\mu = -\omega_{\nu\mu} \tag{21}$$

So $(\omega_1, \omega_2, \ldots, \omega_m, \omega_{m+1}, \ldots, \omega_N)$ form an orthonormal coframe of $\mathcal{F}(\mathbb{E}^N)$. Than the Cartan structure equations on $\mathcal{F}_m(\mathbb{E}^N)$ are:

$$\begin{cases} d\omega_\mu + \displaystyle\sum_{\nu=1}^{N} \omega_{\mu\nu} \wedge \omega_\nu = 0 & \text{(torsion-free)} \\ d\omega_{\mu\nu} + \displaystyle\sum_{\lambda=1}^{N} \omega_{\mu\lambda} \wedge \omega_{\lambda\nu} = 0 & \text{(flat curvature)} \end{cases} \tag{22}$$

Notice that $(\omega_{\mu\nu})$ is the $N \times N$ skew-symmetric matrix connection form of the Lévi-Civita connection on $\mathbb{E}^N$.

[7] The indices i, j and k vary from 1 to m, the indexes a, b and c vary from $m+1$ to N and the indexes μ, ν and λ vary from 1 to N.

Step 3:

Let consider the product manifold $M \times \mathcal{F}_m(\mathbb{E}^N)$. Let $\mathcal{I}_0$ be the exterior ideal on $M \times \mathcal{F}_m(\mathbb{E}^N)$ generated by the Paffafian system $I_0 = \{(\omega_i - \eta_i), \omega_a\}$.

Proposition 4.2.2. *Every m-dimensional integral manifold of $\mathcal{I}_0$ on which the form $\Delta = \omega_1 \wedge \cdots \wedge \omega_m$ does not vanish is locally the graph of a function $f : M \longrightarrow \mathcal{F}_m(\mathbb{E}^N)$ having the property that $u = x \circ f : M \longrightarrow \mathbb{E}^N$ is a local isometric embedding* [8].

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "u"'] & \mathcal{F}_m(\mathbb{E}^N) \arrow[d, "x"] \\ & \mathbb{E}^N \end{tikzcd}$$

Step 4:

According to proposition 4.2.2., the existence of an integral manifold of $\mathcal{I}_0$ for wich Δ is non zero, is a neccessary condition for the existence of a local isometric embedding. However, the theorems and the results that we discussed deal with closed exterior differential system . Therefore it is natural to add to the Pffafian system I_0 the exterior differentiation of each 1-form. We obtain so a closed exterior differential system: $I_0 \cup dI_0$. When we compute the exterior differentiation of $(\omega_i - \eta_i)$, we remark new differential forms and an interesting result,

$$d(\omega_i - \eta_i) = -\sum_{j=1}^{m} (\omega_{ij} - \eta_{ij}) \wedge \omega_i = 0 \tag{23}$$

By Cartan's lemma, $\omega_{ij} - \eta_{ij} = \sum_{k=1}^{m} h_{ijk}\omega_k$, with $h_{ijk} = h_{ikj} = -h_{jik}$. With the symmetry and the skew-symmetry of the functions h_{ijk}, we conclude that h_{ijk} are zero and so, $\omega_{ij} - \eta_{ij} = 0$. This result has a geometric intrepretation: $\omega_{ij} - \eta_{ij} = 0$ implies that $f^*(\omega_{ij}) = \eta_{ij}$ where f is the function of proposition. 4.2.2, which means that the pull-back of Lévi-Civita connection by an isometric embedding is the Lévi-Civita connection on M.

So, we extend the exterior differential I_0 and we obtain an exterior differential system on $M \times \mathcal{F}_m(\mathbb{E}^N)$ $I_1 = \{(\omega_i -$

[8] Conversely, each local isometric embedding $u : M \longrightarrow \mathbb{E}^N$ come uniquely from this construction.

$\eta_i)_{i=1,\ldots,m}, (\omega_a)_{a=m+1,\ldots,N}, (\omega_{ij} - \eta_{ij})_{1\leq i<j\leq m}\}$. In order to have a closed one, we add the exterior differentiation of each form and we get $I = I_1 \cup dI_1$. We denote $\mathcal{I}$ the exterior differential ideal generated by $I = \{(\omega_i - \eta_i), \omega_a, (\omega_{ij} - \eta_{ij}), d(\omega_i - \eta_i), d\omega_a, d(\omega_{ij} - \eta_{ij})\}$.

Instead of looking for integral manifold of $\mathcal{I}_0$, we will look for the existence of an integral manifold of $\mathcal{I}$.

From the structure equations stated earlier, we obtain the following system:

$$\begin{cases} d(\omega_i - \eta_i) \equiv 0 \qquad mod\ I_1 \\ d\omega_a \equiv -\displaystyle\sum_{i=1}^{m} \omega_{ai} \wedge \omega_i \qquad mod\ I_1 \\ d(\omega_{ij} - \eta_{ij}) \equiv \displaystyle\sum_{a=m+1}^{N} \omega_{ai} \wedge \omega_{aj} - \Omega_{ij} \qquad mod\ I_1 \end{cases} \tag{24}$$

On $\mathbb{X}$, the integral manifold of $\mathbb{X}$, $\omega_a = 0$, so $d\omega_a = 0$ too. We conclude that $\displaystyle\sum_{i=1}^{m} \omega_{ai} \wedge \omega_i = 0$. The Cartan lemma (lemma 1.5.1., page 300) ensures the existence of m^2 functions h_{aij} such that $\omega_{ai} = \displaystyle\sum_{j=1}^{m} h_{aij}\omega_j$ where $h_{aij} = h_{aji}$. We can write then: $\omega_{ai} - \displaystyle\sum_{j=1}^{m} h_{aij}\omega_j = 0$ on $\mathbb{X}$.

However, nothing lead us to think that this equality will be true outside $\mathbb{X}$. We define then the differential 1-form π_{ai} on $M \times \mathcal{F}_m(\mathbb{E}^N)$ as follows

$$\pi_{ai} = \omega_{ai} - \sum_{j=1}^{m} h_{aij}\omega_j \tag{25}$$

Consider now, the last equation of the system (24)

$$d(\omega_{ij} - \eta_{ij}) \equiv \sum_{a=m+1}^{N} \omega_{ai} \wedge \omega_{aj} - \Omega_{ij} \qquad mod\ I \tag{26}$$

On $\mathbb{X}$, $\omega_{ij} - \eta_{ij} = 0$, so $d(\omega_{ij} - \eta_{ij}) = 0$. restricted to $\mathbb{X}$, (26) becomes

$$\sum_{a=m+1}^{N} \omega_{ai} \wedge \omega_{aj} = \Omega_{ij}. \tag{27}$$

Using (25), we can write (27) as follows

$$\text{On } \mathbb{X}: \quad \Omega_{ij} = \sum_{k,l=1}^{m} \Big(\sum_{a=m+1}^{N} (h_{aik}h_{ajl} - h_{ail}h_{ajk}) \Big) \omega_k \otimes \omega_l \tag{28}$$

from $\Omega_{ij} = \sum_{k,l=1}^{m} \mathcal{R}_{ijkl}\eta_k \otimes \eta_l = \sum_{k,l=1}^{m} \mathcal{R}_{ijkl}\omega_k \otimes \omega_l$, we conclude that

$$\sum_{a=m+1}^{N} (h_{aik}h_{ajl} - h_{ail}h_{ajk}) = \mathcal{R}_{ijkl} \tag{29}$$

Equation (29) is called the Gauss equation.

We see that the exterior differential system $\tilde{I} = \{(\omega_i - \eta_i), \omega_a, (\omega_{ij} - \eta_{ij}), \pi_{ai}\}$ when the Gauss's equation is satisfied, generates the exterior differential ideal $\mathcal{I}$. Actually, the 1-forms $(\omega_i - \eta_i)$ and ω_a belong to I and to $\tilde{I}$. The 1-forms $(\omega_{ij} - \eta_{ij}) = 0$. This implies that $d(\omega_i - \eta_i) = 0$. The 1-forms $\pi_{ai} = 0$, so $d\omega_a = 0$. From the Gauss equation, $d(\omega_{ij} - \eta_{ij}) = 0$. Looking for integral elements of I is equivalent to looking for integral elements of $\tilde{I}$ for which the Gauss equation is satisfied. We shall proceed this in the following steps. Moreover, $\tilde{I}$ contains less differential 1-form than the exterior differential system I.

Step 5:

The functions h_{aij} are symmetric in their two last indeces. If we consider an $(N-m)$-dimensional euclidean space $\mathcal{W}$, then the matrix (h_{aij}) can be viewed as a symmetric element of $\mathbb{R}^m (i,j = 1,\ldots,m)$ taking value in $\mathcal{W}$, i.e. $(h_{aij}) \in \mathcal{W} \otimes S^2(\mathbb{R}^m)$. Notice that $dim \mathcal{W} \otimes S^2(\mathbb{R}^m) = (N-m)\dfrac{m(m+1)}{2}$.

Proposition 4.2.3. *Let $\mathcal{K}_m$ the set of Riemannian curvature tensors $\mathcal{R}$ such that:*

1. $\mathcal{R}_{ijkl} = \mathcal{R}_{klij}$.
2. $\mathcal{R}_{ijkl} = -\mathcal{R}_{jikl}$.
3. $\mathcal{R}_{ijkl} + \mathcal{R}_{kijl} + \mathcal{R}_{jkil} = 0$.

where the indeces i, j, k and l vary from 1 to m. Then

$$dim \mathcal{K}_m = \frac{m^2(m^2-1)}{12} \tag{30}$$

Lemma 4.2.1. *Suppose that* $r = N - m \geq \dfrac{m(m-1)}{2}$. *Let* $\mathcal{H} \subset \mathcal{W} \otimes S^2(\mathbb{R}^m)$ *an open set containing the elements* $h = (h_{ij})$ *such that the vectors* $\{h_{ij} | 1 \leq i \leq j \leq m-1\}$ *are linearly independents as elements of* $\mathcal{W}$. *The map* $\gamma : \mathcal{H} \longrightarrow \mathcal{K}_m$ *that for* $h \in \mathcal{H}$ *associate* $\gamma(h) \in \mathcal{K}_m$ *such that* $\Big(\gamma(h)\Big)_{ijkl} = \sum_{a=m+1}^{N} h_{aik}h_{ajl} - h_{ail}h_{ajk}$, *is a surjective submersion.*

Step 6: The existence of an m-dimensional ordinary integral element

Let $\mathcal{I}$ the exterior ideal of $M \times \mathcal{F}_m(\mathbb{E}^N)$ generated by $s = N(m+1) - \dfrac{m(m+1)}{2}$ 1-forms:

$$\{\underbrace{(\omega_i - \eta_i)_{i=1,\ldots,m}}_{m}, \underbrace{(\omega_a)_{a=m+1,\ldots N}}_{N-m}, \underbrace{(\omega_{ij} - \eta_{ij})_{1\leq i<j\leq m}}_{\frac{m(m-1)}{2}}, \underbrace{(\pi_{ai})_{i=1,\ldots,m,a=m+1,\ldots N}}_{(N-m)m}\}$$

.

Let $\mathcal{Z} = \{(x, \Upsilon, h) \in M \times \mathcal{F}_m(\mathbb{E}^N) \times \mathcal{H} | \gamma(h) = \mathcal{R}(x)\}$. $\mathcal{Z}$ is a submanifold (the fiber of $\mathcal{R}$ by a submersion. The surjectivity of γ ensure that $\mathcal{Z} \neq \emptyset$). So,

$$\begin{aligned} dim\mathcal{Z} &= dimM + dim\mathcal{F}_m(\mathbb{E}^N) + \dim \mathcal{H} - \dim \mathcal{K}_m \\ &= \underbrace{m}_{dimM} + \underbrace{N(m+1) - \frac{m(m+1)}{2}}_{dim\mathcal{F}_m(\mathbb{E}^N)} + \underbrace{(N-m)\frac{m(m+1)}{2}}_{dim\mathcal{H}} - \underbrace{\frac{m^2(m^2-1)}{12}}_{dim\mathcal{K}_m} \end{aligned} \tag{31}$$

We define the map $\Phi : \mathcal{Z} \longrightarrow \mathcal{V}_m(\mathcal{I}, \Delta)$ that associate to (x, Υ, h) the m-plane at (x, Υ) annihilated by the 1-forms that generate $\mathcal{I}$(the exterior differential system $\tilde{I}$). The map Φ is an embedding and so $\Phi(\mathcal{Z})$ is a submanifold of $\mathcal{V}_m(\mathcal{I}, \Delta)$. We will show that $\Phi(\mathcal{Z})$ contains only ordinary integral elements. In the proof, we will use the proposition.3.2.1.

Let $(x, \Upsilon, h) \in \mathcal{Z}$ be a point. Let denote $E = \Phi(x, \Upsilon, h)$ the integral element defined as follows: $E = \{v \in T_{(x,\Upsilon)}\Big(M \times \mathcal{F}_m(\mathbb{E}^N)\Big) | (\omega_i - \eta_i)(v) = \omega_a(v) = (\omega_{ij} - \eta_{ij})(v) = \pi_{ai}(v) = 0\}$.

E is an m-dimensional integral element. As a matter of fact, s is the number of differential forms that generate the ideal $\mathcal{I}$ and

$$dim\Big(M \times \mathcal{F}_m(\mathbb{E}^N)\Big) - m = N(m+1) - \frac{m(m+1)}{2} = s.$$

We will apply word by word the proposition 3.2.1. Let $\mathcal{I}$ the exterior ideal of $M \times \mathcal{F}_m(\mathbb{E}^N)$ defined above[9]. This ideal does not contain 0-forms. $E \in \mathcal{V}_m(\mathcal{I})$ at $(x, \Upsilon) \in M \times \mathcal{F}_m(\mathbb{E}^N)$. Let $\omega_i, (\omega_i - \eta_i), \omega_a, (\omega_{ij} - \eta_{ij}), \pi_{ai}$ be a coframe[10] of $M \times \mathcal{F}_m(\mathbb{E}^N)$ in the neighborhood of (x, Υ) such that[11] $E = \{v \in T_{x,\Upsilon}\Big(M \times \mathcal{F}_m(\mathbb{E}^N)\Big) | (\omega_i - \eta_i)(v) = \omega_a(v) = (\omega_{ij} - \eta_{ij})(v) = \pi_{ai}(v) = 0\}$.

For $p \leq m$, we define the p-dimensional integral element[12] $E_p = \{x \in E | \omega_k(v) = 0 \text{ pour tout } k > p\}$.[13]. We obtain so, an integral flag $(0)_{(x,\Upsilon)} = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m \subset T_{(x,\Upsilon)}\Big(M \times \mathcal{F}_m(\mathbb{E}^N)\Big)$. We remind that $I = \{\underbrace{(\omega_i - \eta_i), \omega_a, (\omega_{ij} - \eta_{ij})}_{\text{differential 1-forms}}, \underbrace{d(\omega_i - \eta_i), d\omega_a, d(\omega_{ij} - \eta_{ij})}_{\text{differential 2-forms}}\}$..

By computing $d(\omega_i - \eta_j), d\omega_a$ and $d(\omega_{ij} - \eta_{ij})$, we shall find the differential forms that are linear combinations of the forms which generate $\mathcal{I}$.[14]

After simple calculations, we find that

$$d\omega_a \equiv -\sum_{i=1}^{m} \pi_{ai} \wedge \omega_i \tag{32}$$

and

$$d(\omega_{ij} - \eta_{ij}) = \underbrace{\sum_{a=m+1}^{N} \pi_{ai} \wedge \pi_{aj}}_{\blacklozenge} + \sum_{k=1}^{m} \Big(\sum_{a=m+1}^{N} h_{ajk}\pi_{ai} - h_{aik}\pi_{aj} \Big) \wedge \omega_k \tag{33}$$

the term ($\blacklozenge$) is quadratic in π_{ai} and vanishes on $\mathbb{X}$.[15]

[9] $M \times \mathcal{F}_m(\mathbb{E}^N)$ play the role of the manifold "M" in the proposition. 3.2.1.

[10] There is $m + s = dim\Big(M \times \mathcal{F}_m(\mathbb{E}^N)\Big)$ 1-forms.

[11] the $(\omega_i)_{i=1,\ldots,m}$ play the role of "$\omega_1, \omega_2, \ldots, \omega_n$". the $(\omega_i - \eta_i), \omega_a, (\omega_{ij} - \eta_{ij}), \pi_{ai}$ play the role of "π_s" in the proposition 3.2.1.

[12] the exterior differential system I play the role of $\{\varphi_1, \varphi_2, \ldots, \varphi_r\}$" in the proposition 3.2.1.

[13] $E_p \in \mathcal{V}_p(\mathcal{I}, \Delta)$ Because it is annihilated by $s + m - p$ differential 1-forms .

[14] the forms that play the role of π^J_ρ in the proposition 3.2.1.

[15] ($\blacklozenge$) play the role of $\tilde{\varphi}_\rho$ in the proposition 3.2.1.

According to proposition 3.2.1., c_p represente the number of linear independent differential 1-forms.

The differential 1-forms	The indexes	Number of linear independent 1-forms
$\omega_i - \eta_i$	$1 \leq i \leq m$	m
ω_a	$m+1 \leq a \leq N$	$N-m$
$\omega_{ij} - \eta_{ij}$	$1 \leq i < j \leq m$	$\frac{m(m-1)}{2}$
π_{ai}	$1 \leq i \leq p,$ $m+1 \leq a \leq N$	$(N-m)p$
$\sum_{a=m+1}^{N} (h_{aik}\pi_{aj} - h_{ajk}\pi_{ai})$	$1 \leq k \leq p,$ $1 \leq i \leq j \leq m$	$p\frac{(m-p)(m-p-1)}{2} + \frac{p(p+1)}{2}(m-p)$

Finally, by the sum of the number of linear independent 1-forms of the above table, c_p is the codimension of $H(E_p)$ in $G_m\Big(T(M \times \mathcal{F}_m\mathbb{E}^N)\Big)$ defined earlier, and is equal to:

$$c_p = N + \frac{m(m-1)}{2} + (N-m)p + \frac{mp(m-p)}{2} \tag{34}$$

so,

$$\sum_{p=0}^{m-1} c_p = \frac{Nm(m+1)}{2} + \frac{m^2(m^2-1)}{12}. \tag{35}$$

To apply the proposition 3.2.1 and show that E is an m-dimensional ordinary integral element, we need just to compute the codimension of $\Phi(\mathcal{Z})$ in $G_m(\mathcal{I}, \Delta)$.

Let $\mathfrak{U}$ an open set of $\mathcal{F}_m(\mathbb{E}^N)$. So, $dim(M \times \mathfrak{U}) = N(m-1) - \frac{m(m+1)}{2} + m$. We remid that if E is an n-dimensional euclidean space, then the space of all p-planes of E ($G_p(E)$), with $p < n$, is of $p(n-p)$ dimension. Let $(x, \Upsilon) \in M \times \mathfrak{U}$,

$$
\begin{aligned}
&dimG_m\Big(T(M\times\mathfrak{U})\Big) = dimG_m\Big(T_{(x,\Upsilon)}(M\times\mathfrak{U})\Big) + dim(M\times\mathfrak{U})\\
&= m\Big(\underbrace{N(m+1)-\frac{m(m+1)}{2}+m}_{dimT_{(x,\Upsilon)}(M\times\mathfrak{U})=dim(M\times\mathfrak{U})}-m\Big)+\underbrace{N(m+1)-\frac{m(m+1)}{2}+m}_{dim(M\times\mathfrak{U})}\\
&= m\Big(N(m+1)-\frac{m(m+1)}{2}\Big)+N(m+1)-\frac{m(m+1)}{2}+m
\end{aligned}
$$

Since that Φ is an embedding, $dim\Phi(\mathcal{Z}) = dim\mathcal{Z}$, so

$$
\begin{aligned}
&dimG_m\Big(T(M\times\mathfrak{U})\Big) - dim\Phi(\mathcal{Z}) =\\
&\underbrace{Nm(m+1)-\frac{m^2(m+1)}{2}+N(m+1)-\frac{m(m+1)}{2}+m}_{dimG_m\Big(T(M\times\mathfrak{U})\Big)}\\
&\quad\underbrace{-m-N(m+1)+\frac{m(m+1)}{2}-\frac{Nm(m+1)}{2}+\frac{m^2(m+1)}{2}+\frac{m^2(m^2-1)}{12}}_{dim\Phi(\mathcal{Z})}\\
&= \frac{Nm(m+1)}{2}+\frac{m^2(m^2-1)}{12}
\end{aligned}
$$

We conclude that the codimension of $\Phi(\mathcal{Z})$ in $G_m\Big(T(M\times\mathfrak{U})\Big)$ is equal to $c_0 + c_1 + \cdots + c_{m-1}$. By the Cartan's test, $E \in \mathcal{V}_m(\mathcal{I},\Omega)$ is an ordinary integral element of $\mathcal{I}$. The Cartan-Kähler theorem (Corollary 3.3.1.) ensure the existence of an integral manifold $\mathbb{X}$ passing through (x,Υ) and having E as a tangent space at (x,Υ).
$E \in \mathcal{V}_m(\mathcal{I},\Omega)$, In particular, $E \in \mathcal{V}_m(\mathcal{I}_0,\Omega)$. By the proposition 4.2.2. , there exists an isometric embedding of (M,g) in $(\mathbb{E}^N,\varepsilon_N)$.

5 Local Conformal Embedding Problem

Definition 5.0.1. *Let (M,g) and (N,h) be two real analytic Riemannian manifolds of dimension m and n respectively. Let f be a map from M to N. Then f is a conformal embedding from (M,g) to (N,h) if:*

1. *f is a local diffeomorphism;*

2. $f^*h = Sg$, where $S : M \longrightarrow \mathbb{R}^+$ is a stricly positive function on M.

Theorem 5.1. *(Jacobowitch-Moore [5])*
If $dimN = n \geq \frac{1}{2}m(m+1) - 1$, then each point $p \in M$ admit a neighborhood on M which can be conformally embedded in N.

Jacobowitch and Moore gave two differents proofs of this result; one is based on Janet's method and the second on Cartan's method which is close to the proof of Burstin-Cartan-Janet-Schlaefli theorem that we gave.

Roughly speaking, we consider $\mathcal{F}(M) \times \mathcal{F}(N) \times \mathbb{R}^m \times \mathbb{R}^+_*$ and we look for integral manifolds of $I_0 = \{\omega_i - S\eta_i, \omega_a\}$ (the forms are defined as on the previous section). Similarly, we can extend the exterior differential system to obtain a closed one. We lead the deatils of the proof for the reader who should take care of S when he apply the exterior differentiation cause it's a function. When the new exterior differential system is involutive, we look for ordinary integral element and so conclude. (see [5]).

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